

## Electromagnetic and Gravitational Invariants

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2011 J. Phys.: Conf. Ser. 314 012072

(<http://iopscience.iop.org/1742-6596/314/1/012072>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 152.94.0.43

The article was downloaded on 27/10/2011 at 15:15

Please note that [terms and conditions apply](#).

# Electromagnetic and Gravitational Invariants

L. Filipe O. Costa\*, Carlos A. R. Herdeiro<sup>†</sup>, Lode Wylleman<sup>‡</sup>

\*Centro de Física do Porto, Universidade do Porto, Porto, Portugal

<sup>†</sup>Dep. de Física da Universidade de Aveiro and I3N, 3810-183 Aveiro, Portugal

<sup>‡</sup>Faculty of App. Sciences TW16, Ghent University, Galglaan 2, 9000 Ghent, Belgium

E-mail: filipezola@fc.up.pt, herdeiro@ua.pt, lwyllema@gmail.com

**Abstract.** The curvature invariants have been subject of recent interest in the context of the experimental detection of the gravitomagnetic field, namely due to the debate concerning the notions of “extrinsic” and “intrinsic” gravitomagnetism. In this work we explore the physical meaning of the curvature invariants, dissecting their relationship with the gravitomagnetic effects.

## 1. Introduction

There is an ongoing debate concerning the question if there is a fundamental difference between the “translational” gravitomagnetism claimed to have been detected with Lunar Laser Ranging [1], and (to an high accuracy) in the observations [2] of the binary pulsar PSR 1913 +16, and the gravitomagnetic field produced by the rotation of the Earth, detected in the LAGEOS Satellites data [3] (whose detection was also the primary goal of the Gravity Probe-B mission). These phenomena have been dubbed, respectively, “extrinsic” and “intrinsic” gravitomagnetism, such distinction being based [4, 5] on the quadratic curvature invariants  $\mathbf{R} \cdot \mathbf{R}$ ,  $\star \mathbf{R} \cdot \mathbf{R}$ , and driven by their formal analogy with the invariants  $\mathbf{F} \cdot \mathbf{F}$ ,  $\star \mathbf{F} \cdot \mathbf{F}$  of the Maxwell tensor. In this work [6], starting from previous knowledge on the classification of purely electric/magnetic spacetimes [7, 8], and using insight from the *formal* analogy  $\{\mathbf{F} \cdot \mathbf{F}, \star \mathbf{F} \cdot \mathbf{F}\} \leftrightarrow \{\mathbf{R} \cdot \mathbf{R}, \star \mathbf{R} \cdot \mathbf{R}\}$ , we explain the invariant structure of the relevant gravitational fields. Then, using the gravito-electromagnetic analogy based on tidal tensors [9] as a *physical* guiding principle, we clarify the physical meaning of the curvature invariants and the implications on the motion of test particles.

## 2. Electromagnetic and gravitational scalar invariants — algebraic meaning

Wrt any unit timelike 4-vector  $u^\alpha$ , the Maxwell tensor splits irreducibly into the two spatial vectors  $(E^u)^\alpha \equiv F^{\alpha\beta}u_\beta$  and  $(B^u)^\alpha \equiv \star F^{\alpha\beta}u_\beta$  (c.f. Eq. (16) of [10]), which are, respectively, the electric and magnetic fields as measured by an observer of 4-velocity  $u^\alpha$ . Both  $(E^u)^\alpha$  and  $(B^u)^\alpha$  depend on the observer 4-velocity  $u^\alpha$ , but combining them

one can construct the two quadratic (i.e., second order) scalar invariants (e.g. [11]):

$$E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} \equiv -\frac{1}{2}\star\mathbf{F}\cdot\mathbf{F}; \quad E^\alpha B_\alpha = -\frac{1}{4}F_{\alpha\beta}\star F^{\alpha\beta} \equiv -\frac{1}{4}\star\mathbf{F}\cdot\mathbf{F}. \quad (1)$$

They have the following physical interpretation: (i) if  $\star\mathbf{F}\cdot\mathbf{F} \neq 0$  then for all observers  $u^\alpha$  one has  $(E^u)^\alpha \neq 0 \neq (B^u)^\alpha$ ; (ii) if  $\star\mathbf{F}\cdot\mathbf{F} = 0$  and  $\mathbf{F}\cdot\mathbf{F} > 0$  ( $< 0$ ) then observers  $u^\alpha$  do exist wrt  $(B^u)^\alpha = 0$  ( $(E^u)^\alpha = 0$ ); (iii) the case  $\mathbf{F}\cdot\mathbf{F} = \star\mathbf{F}\cdot\mathbf{F} = 0$  (but  $F^{\alpha\beta} \neq 0$ ) corresponds to pure electromagnetic radiation.

On the other hand, the Riemann tensor *in vacuum* becomes the Weyl tensor, which analogously is completely decomposed (c.f. Eq. (30) of [10]) in its electric part  $\mathbb{E}_{\alpha\beta} = R_{\alpha\mu\beta\nu}u^\mu u^\nu$  and magnetic part  $\mathbb{H}_{\alpha\beta} = \star R_{\alpha\mu\beta\nu}u^\mu u^\nu$  wrt  $u^\alpha$ . Moreover,  $\mathbb{E}_{\alpha\beta}$  and  $\mathbb{H}_{\alpha\beta}$  form quadratic invariants *formally* analogous to (1):

$$\mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{8}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \equiv \frac{1}{8}\mathbf{R}\cdot\mathbf{R} \quad \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{16}R_{\alpha\beta\gamma\delta}\star R^{\alpha\beta\gamma\delta} \equiv \frac{1}{16}\star\mathbf{R}\cdot\mathbf{R}, \quad (2)$$

which are usually combined into  $I \equiv (\mathbf{R}\cdot\mathbf{R} + i\star\mathbf{R}\cdot\mathbf{R})/8$ . However, whereas the invariants (1) are the only two independent scalar invariants of  $F^{\alpha\beta}$ , in the case of  $R_{\alpha\beta\gamma\delta}$  there also two independent *cubic* invariants, given by  $A \equiv R_{\lambda\mu}^{\alpha\beta}R^{\lambda\mu}_{\rho\sigma}R^{\rho\sigma}_{\alpha\beta}/16$  and  $B \equiv R_{\lambda\mu}^{\alpha\beta}R^{\lambda\mu}_{\rho\sigma}\star R^{\rho\sigma}_{\alpha\beta}/16$ , and combined into  $J \equiv A - iB$ . It turns out [7, 8] that one basically obtains formally equivalent statements (i)-(iii) as above by replacing  $\mathbf{F}$  by  $\mathbf{R}$ , *provided that the condition*  $M \equiv I^3/J^2 - 6 \geq 0$  (real or infinite) *is added to* (ii). Further comments and a more detailed treatment hereof shall be given in [6].

### 3. Interpretation of the invariant's structure for relevant setups

For astrophysical applications we will be interested essentially in the gravitational far field of non-spinning and spinning bodies obeying criterion  $M \geq 0$ , which thus have an electromagnetic counterpart. Now the important point is that  $\mathbb{E}_{\alpha\gamma}$  and  $\mathbb{H}_{\alpha\gamma}$  also obey transformation laws in a change of observer/frame which exhibit a degree of similarity with the transformation of the electromagnetic fields that allows for invariant structure of a gravitational setup to be understood comparing with the analogous electromagnetic setup. The magnetic part of the Riemann tensor  $(\mathbb{H}^U)_{\alpha\gamma}$  as measured by a given observer  $U^\alpha = U^0(1, \vec{v})$ , can be obtained, in terms of the tensors  $(\mathbb{E}^u)_{\alpha\gamma}$  and  $(\mathbb{H}^u)_{\alpha\gamma}$  measured by  $u^\alpha = u^0\delta_0^\alpha$ , from decomposition (30) of [10], yielding [13], to first order in  $v$ , Eq. (3i) below, which exhibits suggestive similarities with its electromagnetic counterpart (3ii):

$$\overleftrightarrow{\mathbb{H}}(U) \simeq \overleftrightarrow{\mathbb{E}}(u) \times \vec{v} - \vec{v} \times \overleftrightarrow{\mathbb{E}}(u) + \overleftrightarrow{\mathbb{H}}(u) \quad (i) \quad \vec{B}(U) \simeq -\vec{v} \times \vec{E}(u) + \vec{B}(u) \quad (ii) \quad (3)$$

where  $(\overleftrightarrow{\mathbb{E}} \times \vec{v})_{kl} \equiv \epsilon^{ij}_l \mathbb{E}_{ik} v_j$ ,  $(\vec{v} \times \overleftrightarrow{\mathbb{E}})_{kl} \equiv \epsilon^{ij}_k v_i \mathbb{E}_{lj}$ .

The examples of interest will be discussed in detail in [6]. Consider first a single point charge; for a generic observer  $U^\alpha$ , we have  $\vec{B}(U) \neq 0$ ; but as one can read from (3ii), always  $\vec{B} \perp \vec{E}$ , since there are observers (those comoving with the charge) for which  $\vec{B}$  vanishes everywhere; and the invariant  $\star\mathbf{F}\cdot\mathbf{F} = 0$  reflects that. Analogous arguments explain the vanishing of  $\star\mathbf{R}\cdot\mathbf{R} = 0$  in Schwarzschild spacetime.

If we consider a system of two charges in arbitrary motion, then in general  $\star\mathbf{F} \cdot \mathbf{F} \neq 0$ , there is no frame where  $\vec{B} = 0$  *globally*, which we relate with the fact that there is no observer for which *both* particles are at rest (i.e., the currents involved cannot be made to vanish by a Lorentz boost). However, as explicitly shown in [6], in the special case of planar motion, *in the plane of the motion* we have  $\star\mathbf{F} \cdot \mathbf{F} = 0$ , meaning that *therein* (and only therein) there are observers, whose 4-velocity  $U^\alpha$  varies from point to point, relative to which  $\vec{B}$  vanishes, which arises from an exact cancellation between the magnetic field produced by the motion of the two charges relative to the observer  $U^\alpha$ . It should be emphasized that even in the plane of the motion  $\vec{B}$  does not globally vanish in any rigid frame (as the velocity field  $U^\alpha$  for which  $B^\alpha = 0$  is a shearing congruence), which is a clearly distinct situation from the case of a single charge, despite both being characterized by  $\star\mathbf{F} \cdot \mathbf{F} \neq 0$ . As shall be shown in [6], the analogous gravitational two body systems (which are studied to Post Newtonian order) exhibit an analogous invariant structure:  $\star\mathbf{R} \cdot \mathbf{R} \neq 0$  in the most general case, telling us that in general there are no observers for which  $\mathbb{H}_{\alpha\beta} = 0$ ; and for planar motion, which is the case of the Earth-Sun system discussed in [1] we have  $\star\mathbf{R} \cdot \mathbf{R} = 0$  in the orbital plane, and  $\star\mathbf{R} \cdot \mathbf{R} \neq 0$  elsewhere. This structure can be explained by precisely the same reasoning as in the electromagnetic case, since the tensor  $\mathbb{H}_{\alpha\beta}$  is linear to Post Newtonian order and therefore to this accuracy a superposition principle applies.

One of the key points of this work is that the very same arguments are shown to apply to the case of a spinning body. The structure of the electromagnetic invariants of the field of a spinning charge is similar to the one of a system of charges in planar motion:  $\star\mathbf{F} \cdot \mathbf{F} = 0$  in the equatorial plane,  $\star\mathbf{F} \cdot \mathbf{F} \neq 0$  elsewhere (explicit expressions will be given in [6]). A spinning body can be viewed as an assembly of translating elements; and there is no observer for which every part of it is at rest<sup>1</sup>; hence (except in the equatorial plane) it is not possible to make  $B^\alpha$  vanish. In the equatorial plane, again by exactly the same principle from the case of  $N \geq 2$  sources in planar motion, there are observers for which  $B^\alpha = 0$  (*not* the observers co-rotating with the source, despite rotating in the same sense; these observers have a velocity that decays as  $r^{-1}$ , which is the same asymptotic dependence as case of the two body system in planar motion mentioned above, c.f. [6]). The analogy with the gravitational case again holds: the curvature invariants for Kerr spacetime have the structure  $\star\mathbf{R} \cdot \mathbf{R} \neq 0$  elsewhere, and  $\star\mathbf{R} \cdot \mathbf{R} = 0$  (exactly!) in the equatorial plane, where, likewise, there are observers for which  $\mathbb{H}_{\alpha\beta} = 0$ ; their angular velocity (which again depends on  $r$ ) asymptotically matches its electromagnetic counterpart up to a factor of 2 [6, 12]. Hence, from the point of view of the structure of the invariants, whilst clearly distinguishing between the fields produced by a *single* translating body from the ones of a rotating body (which is in agreement with [5]), we found no fundamental distinction between the latter and the

<sup>1</sup> In a co-rotating frame indeed the whole spinning body would be seen to be at “rest”; but a (rigid) rotating frame consists of a congruence of observers *all with different* 4-velocities  $U^\alpha$ , whilst having the same angular velocity.  $\vec{B}$  does *not* vanish in the co-rotating frame, even though there are no currents therein; the reason being (taking the perspective of the rotating frame) that the vorticity of the congruence also contributes as a source for  $\vec{B}$  (see Eqs. (17)-(20) of [10]). It is crucial to realize that unlike the translation of a single body, which can always be made to vanish by a simple boost, rotation (like the motion of  $N \geq 2$  translating bodies in general) imprints itself in intrinsic properties (like the intrinsic angular momentum) that cannot be made to vanish by any change of frame.

ones from a system of translating bodies in planar motion. This is one point that (we hope) may shed some light on the ongoing debate.

It should also be remarked that, as the examples above make clear, whereas  $\star\mathbf{F}\cdot\mathbf{F}\neq 0$  ( $\star\mathbf{R}\cdot\mathbf{R}\neq 0$ ) signals intrinsic magnetic field (curvature), the case  $\star\mathbf{F}\cdot\mathbf{F}=0$  ( $\star\mathbf{R}\cdot\mathbf{R}=0$  with  $M\geq 0$ ) does not distinguish static solutions, e.g. Coulomb field of a point charge (Schwarzschild solution), where  $B^\alpha(\mathbb{H}_{\alpha\beta})$  vanishes globally in a rigid frame, from setups where  $B^\alpha(\mathbb{H}_{\alpha\beta})$  can only be made to vanish for a class of observers specific to each point (i.e., forming shearing congruences), as is the case of the equatorial plane of a spinning charge (equatorial plane of Kerr spacetime).

#### 4. Dynamical implications of the invariants. Gravitomagnetism.

In spite of the insight it provides to interpret the structure of the gravitational invariants, the analogy with the electromagnetic invariants is purely formal (see Sec. 4 of [14]), and should not be used as a physical guiding principle: in one case we are dealing with quantities built on electromagnetic *fields*  $E^\alpha$ ,  $B^\alpha$ ; in the other case with gravitational *tidal tensors*  $\mathbb{E}_{\alpha\beta}$ ,  $\mathbb{H}_{\alpha\beta}$ , which, as shown in [9], play in gravity analogous dynamical roles *not* to  $E^\alpha$ ,  $B^\alpha$ , but to the *electromagnetic tidal tensors* (which are quantities one order higher in differentiation). This distinction is crucial, because the effects involved may actually be opposite:  $\star\mathbf{F}\cdot\mathbf{F}=0$ , telling us that there are observers for which  $B^\alpha=0$ , means that magnetic dipoles carried by them do not undergo Larmor precession, but in general they will feel a force (since the magnetic *tidal* field does not vanish [12, 6]); by contrast what  $\star\mathbf{R}\cdot\mathbf{R}=0$  (with  $M\geq 0$ ) tells us is that there is a class of observers carrying gyroscopes that do not feel gravitational force (see Eq. (28) of [9]), but in general they “precess” (since it is the gravitomagnetic *tidal tensor*  $\mathbb{H}_{\alpha\beta}$  that vanishes, not the so called [1, 4, 5] “gravitomagnetic *field*”). In [6] this will be shown explicitly for the physical systems considered in the previous section. To conclude, curvature invariants *do not* tell us about the “gravitomagnetic *field*” itself, nor the Lense Thirring effect (gravitomagnetic field and gyroscope “precession” are actually artifacts of the reference frame, thus cannot be manifest in invariant quantities). Curvature invariants tell us about the gravitomagnetic *tidal* field (magnetic curvature), and the appropriate probe to measure it is the *force* (*not the precession!*) on a gyroscope.

#### References

- [1] M. Soffel *et al*, *PRD* **78** (2008) 024033; T. W. Murphy *et al*, *PRL* **98** (2007) 071102
- [2] K. Nordtvedt, *Int. J. Theoretical Physics* **27** (1988) 2347
- [3] I. Ciufolini, E. Pavlis, *Nature* **431** (2004) 958;
- [4] I. Ciufolini, *New Astronomy* **15** (2010) 332
- [5] I. Ciufolini, J. A. Wheeler, “Gravitation and Inertia,” Princeton Ser. in Phys. (1995); Sec. 6.11.
- [6] Costa L. F. O., Herdeiro C. A. R., Wylleman L. (work in progress)
- [7] C. McIntosh, R. Arianhod, S. T. Wade, C. Hoenselaers, *Class. Quant. Grav.* **11** (1994) 1555
- [8] L. Wylleman, N. Van den Bergh, *Phys. Rev. D.* **74** (2006) 084001
- [9] Costa L F O and Herdeiro C A R 2008 *Phys. Rev. D* **78** 024021
- [10] R. Maartens, B. Bassett, *Class. Q. Grav.* **15** (1998) 705
- [11] L. Bel, *Cahiers de Physique*, **16** (1962) 59; eng. trans. *Gen. Rel. Grav.*, **32** (2000) 2047.
- [12] Costa L F O, Herdeiro C A R, Natário J, Zilhão M, in preparation.
- [13] W. Campbell, T. Morgan, *Am. J. Phys.* **44** (1976) 356.
- [14] L. F. Costa, C. A. R. Herdeiro, [arXiv:gr-qc/0612140v2]